GROTHENDIECK GROUP AND GENERALIZED MUTATION RULE FOR 2-CALABI–YAU TRIANGULATED CATEGORIES

YANN PALU

Abstract. We compute the Grothendieck group of certain 2-Calabi–Yau triangulated categories appearing naturally in the study of the link between quiver representations and Fomin–Zelevinsky's cluster algebras. In this setup, we also prove a generalization of Fomin–Zelevinsky's mutation rule.

Introduction

In their study [6] of the connections between cluster algebras (see [22]) and quiver representations, P. Caldero and B. Keller conjectured that a certain antisymmetric bilinear form is well–defined on the Grothendieck group of a cluster–tilted algebra associated with a finite–dimensional hereditary algebra. The conjecture was proved in [19] in the more general context of Hom-finite 2-Calabi–Yau triangulated categories. It was used in order to study the existence of a cluster character on such a category $\mathcal{C}$, by using a formula proposed by Caldero–Keller.

In the present paper, we restrict to the case where $\mathcal{C}$ is algebraic (i.e. is the stable category of a Frobenius category). We first use this bilinear form to prove a generalized mutation rule for quivers of cluster–tilting subcategories in $\mathcal{C}$. When the cluster–tilting subcategories are related by a single mutation, this shows, via the method of [9], that their quivers are related by the Fomin–Zelevinsky mutation rule. This special case was already proved in [3], without assuming $\mathcal{C}$ to be algebraic.

We also compute the Grothendieck group of the triangulated category $\mathcal{C}$. In particular, this allows us to improve on results by M. Barot, D. Kussin and H. Lenzing: We compare the Grothendieck group of a cluster category $\mathcal{C}_A$ with the group $\overline{\mathrm{K}}_0(\mathcal{C}_A)$. The latter group was defined in [1] by only considering the triangles in $\mathcal{C}_A$ which are induced by those of the derived category. More precisely, we prove that those two groups are isomorphic for any cluster category associated with a finite dimensional hereditary algebra, with its triangulated structure defined by B. Keller in [16].

This paper is organized as follows: The first section is dedicated to notation and necessary background from [8], [9], [17], [19]. In section 2, we compute the Grothendieck group of the triangulated category $\mathcal{C}$. In section 3, we prove a generalized mutation rule for quivers of cluster–tilting subcategories in $\mathcal{C}$. In particular, this yields a new proof of the Fomin–Zelevinsky mutation rule, under the restriction that $\mathcal{C}$ is algebraic. We finally show that $\mathrm{K}_0(\mathcal{C}_A) = \overline{\mathrm{K}}_0(\mathcal{C}_A)$ for any finite dimensional hereditary algebra A.

Acknowledgements

This article is part of my PhD thesis, under the supervision of Professor B. Keller. I would like to thank him deeply for introducing me to the subject and for his infinite patience.

Contents

1. Notations and background

Let $\mathcal{E}$ be a Frobenius category whose idempotents split and which is linear over a given algebraically closed field k. By a result of Happel [10], its stable category $\mathcal{C} = \underline{\mathcal{E}}$ is triangulated. We assume moreover, that $\mathcal{C}$ is Hom-finite, 2-Calabi–Yau and has a cluster–tilting subcategory (see section 1.2), and we denote by Σ its suspension functor. Note that we do not assume that $\mathcal{E}$ is Hom-finite.

We write $\mathcal{X}(\ ,\ )$, or $\mathrm{Hom}_{\mathcal{X}}(\ ,\ )$, for the morphisms in a category $\mathcal{X}$ and $\mathrm{Hom}_{\mathcal{X}}(\ ,\ )$ for the morphisms in the category of $\mathcal{X}$-modules. We also denote by $X^\wedge$ the projective $\mathcal{X}$-module represented by X: $X^\wedge = \mathcal{X}(?, X)$.

1.1. Fomin–Zelevinsky mutation for matrices.

Let $B = (b_{ij})_{i,j\in I}$ be a finite or infinite matrix, and let k be in I. The Fomin and Zelevinsky mutation of B (see [8]) in direction k is the matrix

$$\mu_k(B) = (b'_{ij})$$

defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{else.} \end{cases}$$

Note that $\mu_k\big(\mu_k(B)\big) = B$ and that if B is skew-symmetric, then so is $\mu_k(B)$.

We recall two lemmas of [9], stated for infinite matrices, which will be useful in section 3. Note that lemma 7.2 is a restatement of [2, (3.2)]. Let $S = (s_{ij})$ be the matrix defined by

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| - b_{ij}}{2} & \text{if } i = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

Lemma 7.1 ([9, Geiss–Leclerc–Schröer]) : *Assume that B is skew-symmetric. Then, $S^2 = 1$ and the (i, j)-entry of the transpose of the matrix S is given by*

$$s^{\mathrm{t}}_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| + b_{ij}}{2} & \text{if } j = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

The matrix S yields a convienent way to describe the mutation of B in the direction k:

Lemma 7.2 ([9, Geiss–Leclerc–Schröer], [2, Berenstein–Fomin–Zelevinsky]**)** : *Assume that B is skew-symmetric. Then we have:*

$$\mu_k(B) = S^{\mathrm{t}} B S.$$

Note that the product is well-defined since the matrix S has a finite number of non vanishing entries in each column.

1.2. **Cluster–tilting subcategories.** A cluster–tilting subcategory (see [17]) of $\mathcal{C}$ is a full subcategory $\mathcal{T}$ such that

a) $\mathcal{T}$ is a linear subcategory;
b) for any object X in $\mathcal{C}$, the contravariant functor $\mathcal{C}(?, X)|_{\mathcal{T}}$ is finitely generated;
c) for any object X in $\mathcal{C}$, we have $\mathcal{C}(X, \Sigma T) = 0$ for all T in $\mathcal{T}$ if and only if X belongs to $\mathcal{T}$.

We now recall some results from [17], which we will use in the sequel. Let $\mathcal{T}$ be a cluster–tilting subcategory of $\mathcal{C}$, and denote by $\mathcal{M}$ its preimage in $\mathcal{E}$. In particular $\mathcal{M}$ contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective-injective objects, and we have $\underline{\mathcal{M}} = \mathcal{T}$.

The following proposition will be used implicitly, extensively in this paper.
Proposition [17, Keller–Reiten] :

a) *The category* $\operatorname{mod} \underline{\mathcal{M}}$ *of finitely presented* $\underline{\mathcal{M}}$*-modules is abelian.*
b) *For each object $X \in \mathcal{C}$, there is a triangle*

$$\Sigma^{-1} X \longrightarrow T_1^X \longrightarrow T_0^X \longrightarrow X$$

of $\mathcal{C}$, with T_0^X and T_1^X in $\mathcal{T}$.

Recall that the perfect derived category $\operatorname{per} \mathcal{M}$ is the full triangulated subcategory of the derived category of $\mathcal{D} \operatorname{Mod} \mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules.
Proposition [17, Keller–Reiten] :

a) *For each $X \in \mathcal{E}$, there are conflations*

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow X \longrightarrow M^0 \longrightarrow M^1 \longrightarrow 0$$

in $\mathcal{E}$, with M_0, M_1, M^0 and M^1 in $\mathcal{M}$.
b) *Let Z be in* $\operatorname{mod} \underline{\mathcal{M}}$. *Then Z considered as an $\mathcal{M}$-module lies in the perfect derived category* $\operatorname{per} \mathcal{M}$ *and we have canonical isomorphisms*

$$D(\operatorname{per} \mathcal{M})(Z, ?) \simeq (\operatorname{per} \mathcal{M})(?, Z[3]).$$

1.3. **The antisymmetric bilinear form.** In section 3, we will use the existence of the antisymmetric bilinear form $\langle \ , \ \rangle_a$ on $\mathrm{K}_0(\operatorname{mod} \underline{\mathcal{M}})$. We thus recall its definition from [6].

Let $\langle \ , \ \rangle$ be a truncated Euler form on $\operatorname{mod} \underline{\mathcal{M}}$ defined by

$$\langle M, N \rangle = \dim \operatorname{Hom}_{\underline{\mathcal{M}}}(M, N) - \dim \operatorname{Ext}^1_{\underline{\mathcal{M}}}(M, N)$$

for any $M, N \in \operatorname{mod} \underline{\mathcal{M}}$. Define $\langle \ , \ \rangle_a$ to be the antisymmetrization of this form:

$$\langle M, N \rangle_a = \langle M, N \rangle - \langle N, M \rangle.$$

This bilinear form descends to the Grothendieck group $\mathrm{K}_0(\operatorname{mod} \underline{\mathcal{M}})$:
Lemma [19, section 3] : The antisymmetric bilinear form

$$\langle M, N \rangle_a : \mathrm{K}_0(\operatorname{mod} \underline{\mathcal{M}}) \times \mathrm{K}_0(\operatorname{mod} \underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$$

is well-defined.

2. Grothendieck groups of algebraic 2-CY categories with a cluster–tilting subcategory

We fix a cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{C}$, and we denote by $\mathcal{M}$ its preimage in $\mathcal{E}$. In particular $\mathcal{M}$ contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective-injective objects, and we have $\underline{\mathcal{M}} = \mathcal{T}$.

We denote by $\mathcal{H}^b(\mathcal{E})$ and $\mathcal{D}^b(\mathcal{E})$ respectively the bounded homotopy category and the bounded derived category of $\mathcal{E}$. We also denote by $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$, $\mathcal{H}^b(\mathcal{P})$, $\mathcal{H}^b(\mathcal{M})$ and $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ the full subcategories of $\mathcal{H}^b(\mathcal{E})$ whose objects are the $\mathcal{E}$-acyclic complexes, the complexes of projective objects in $\mathcal{E}$, the complexes of objects of $\mathcal{M}$ and the $\mathcal{E}$-acyclic complexes of objects of $\mathcal{M}$, respectively.

2.1. **A short exact sequence of triangulated categories.**

Lemma 1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be thick, full triangulated subcategories of a triangulated category $\mathcal{A}$ and let $\mathcal{B}$ be $\mathcal{A}_1 \cap \mathcal{A}_2$. Assume that for any object X in $\mathcal{A}$ there is a triangle $X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow \Sigma X_1$ in $\mathcal{A}$, with X_1 in $\mathcal{A}_1$ and X_2 in $\mathcal{A}_2$. Then the induced functor $\mathcal{A}_1/\mathcal{B} \longrightarrow \mathcal{A}/\mathcal{A}_2$ is a triangle equivalence.*

Proof. Under these assumptions, denote by F the induced triangle functor from $\mathcal{A}_1/\mathcal{B}$ to $\mathcal{A}/\mathcal{A}_2$. We are going to show that the functor F is a full, conservative, dense functor. Since any full conservative triangle functor is fully faithful, F will then be an equivalence of categories.

We first show that F is full. Let X_1 and X_1' be two objects in $\mathcal{A}_1$. Let f be a morphism from X_1 to X_1' in $\mathcal{A}/\mathcal{A}_2$ and let

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, "w"] & \\ X_1 & & X_1' \end{tikzcd}$$

be a left fraction which represents f. The morphism w is in the multiplicative system associated with $\mathcal{A}_2$ and thus yields a triangle $\Sigma^{-1}A_2 \to Y \xrightarrow{w} X_1' \to A_2$ where A_2 lies in the subcategory $\mathcal{A}_2$. Moreover, by assumption, there exists a triangle $Y_1 \to Y \to Y_2 \to \Sigma Y_1$ with Y_i in $\mathcal{A}_i$. Applying the octahedral axiom to the composition $Y_1 \to Y \to X_1'$ yields a commutative diagram whose two middle rows and columns are triangles in $\mathcal{A}$

$$\begin{tikzcd} & \Sigma^{-1}A_2 \arrow[r, equal] \arrow[d] & \Sigma^{-1}A_2 \arrow[d] & \\ Y_1 \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & \Sigma Y_1 \arrow[d, equal] \\ Y_1 \arrow[r] & X_1' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma Y_1 \\ & A_2 \arrow[r, equal] & A_2 & \end{tikzcd}.$$

Since Y_2 and A_2 belong to $\mathcal{A}_2$, so does Z. And since X_1' and Y_1 belong to $\mathcal{A}_1$, so does Z. This implies, that Z belongs to $\mathcal{B}$. The morphism $Y_1 \to X_1'$ is in the multiplicative system of $\mathcal{A}_1$ associated with $\mathcal{B}$ and the diagram

$$\begin{tikzcd} & Y_1 \arrow[dl] \arrow[dr] & \\ X_1 & & X_1' \end{tikzcd}$$

is a left fraction which represents f. This implies that f is the image of a morphism in $\mathcal{A}_1/\mathcal{B}$. Therefore the functor F is full.

We now show that F is conservative. Let $X_1 \xrightarrow{f} Y_1 \to Z_1 \to \Sigma X_1$ be a triangle in $\mathcal{A}_1$. Assume that Ff is an isomorphism in $\mathcal{A}/\mathcal{A}_2$, which implies that Z_1 is an object of $\mathcal{A}_2$. Therefore, Z_1 is an object of $\mathcal{B}$ and f is an isomorphism in $\mathcal{A}_1/\mathcal{B}$.

We finally show that F is dense. Let X be an object of the category $\mathcal{A}/\mathcal{A}_2$, and let $X_1 \to X \to X_2 \to \Sigma X_1$ be a triangle in $\mathcal{A}$ with X_i in $\mathcal{A}_i$. Since X_2 belongs to $\mathcal{A}_2$, the image of the morphism $X_1 \to X$ in $\mathcal{A}/\mathcal{A}_2$ is an isomorphism. Thus X is isomorphic to the image by F of an object in $\mathcal{A}_1/\mathcal{B}$. $\square$

As a corollary, we have the following:

Lemma 2. *The following sequence of triangulated categories is short exact:*

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M}) \longrightarrow \mathcal{D}^b(\mathcal{E}) \longrightarrow 0.$$

Remark: This lemma remains true if $\mathcal{C}$ is d-Calabi–Yau and $\underline{\mathcal{M}}$ is $(d-1)$-cluster–tilting, using section 5.4 of [17].

Proof. For any object X in $\mathcal{H}^b(\mathcal{E})$, the existence of an object M in $\mathcal{H}^b(\mathcal{M})$ and of a quasi-isomorphism w from M to X is obtained using the approximation conflations of Keller–Reiten (see section 1.2). Since the cone of the morphism w belongs to $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$, lemma 1 applies to the subcategories $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$, $\mathcal{H}^b(\mathcal{M})$ and $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$ of $\mathcal{H}^b(\mathcal{E})$. $\square$

Proposition 3. *The following diagram is commutative with exact rows and columns:*

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \arrow[r, "i_{\mathcal{M}}"] & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r] \arrow[u] & \underline{\mathcal{E}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \arrow[r] \arrow[u, equal] & \mathcal{H}^b(\mathcal{M}) \arrow[r] \arrow[u] & \mathcal{D}^b(\mathcal{E}) \arrow[r] \arrow[u] & 0 \\
 & & \mathcal{H}^b(\mathcal{P}) \arrow[r, equal] \arrow[u] & \mathcal{H}^b(\mathcal{P}) \arrow[r] \arrow[u, "i_{\mathcal{P}}"'] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \qquad (D)$$

Proof. The column on the right side has been shown to be exact in [18] and [20]. The second row is exact by lemma 2. The subcategories $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ and $\mathcal{H}^b(\mathcal{P})$ of $\mathcal{H}^b(\mathcal{M})$ are left and right orthogonal to each other. This implies that the induced functors $i_{\mathcal{M}}$ and $i_{\mathcal{P}}$ are fully faithful and that taking the quotient of $\mathcal{H}^b(\mathcal{M})$ by those two subcategories either in one order or in the other gives the same category. Therefore the first row is exact. $\square$

2.2. **Invariance under mutation.** A natural question is then to which extent the diagram (D) depends on the choice of a particular cluster–tilting subcategory. Let thus $\mathcal{T}'$ be another cluster–tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be its preimage in $\mathcal{E}$. Let $\operatorname{Mod}\mathcal{M}$ (resp. $\operatorname{Mod}\mathcal{M}'$) be the category of $\mathcal{M}$-modules (resp. $\mathcal{M}'$-modules), i.e. of k-linear contravariant functors from $\mathcal{M}$ (resp. $\mathcal{M}'$) to the category of k-vector spaces.

Let X be the $\mathcal{M}$-$\mathcal{M}'$-bimodule which sends the pair of objects (M, M') to the k-vector space $\mathcal{E}(M, M')$. The bimodule X induces a functor $F = ? \otimes_{\mathcal{M}'} X : \operatorname{Mod}\mathcal{M}' \longrightarrow \operatorname{Mod}\mathcal{M}$ denoted by T_X in [15, section 6.1].

Recall that the perfect derived category $\operatorname{per}\mathcal{M}$ is the full triangulated subcategory of the derived category $\mathcal{D}\operatorname{Mod}\mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules.

Proposition 4. *The left derived functor*

$$\mathbb{L}F: \quad \mathcal{D}\operatorname{Mod}\mathcal{M}' \quad \longrightarrow \quad \mathcal{D}\operatorname{Mod}\mathcal{M}$$

is an equivalence of categories.

Proof. Recall that if X is an object in a category $\mathcal{X}$, we denote by $X\hat{}$ the functor $\mathcal{X}(?,X)$ represented by X. By [15, 6.1], it is enough to check the following three properties:

1. For all objects M', M'' of $\mathcal{M}$, the group $\operatorname{Hom}_{\mathcal{D}\operatorname{Mod}\mathcal{M}}(\mathbb{L}FM'\hat{}, \mathbb{L}FM''\hat{}[n])$ vanishes for $n \neq 0$ and identifies with $\operatorname{Hom}_{\mathcal{M}'}(M', M'')$ for $n = 0$;
2. for any object M' of $\mathcal{M}'$, the complex $\mathbb{L}FM'\hat{}$ belongs to $\operatorname{per}\mathcal{M}$;
3. the set $\{\mathbb{L}FM'\hat{},\, M' \in \mathcal{M}'\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}$ as a triangulated category with infinite sums.

Let M' be an object of $\mathcal{M}'$, and let $M_1 \rightarrowtail M_0 \twoheadrightarrow M'$ be a conflation in $\mathcal{E}$, with M_0 and M_1 in $\mathcal{M}$, and whose deflation is a right $\mathcal{M}$-approximation (c.f. section 4 of [17]). The surjectivity of the map $M_0\hat{} \longrightarrow \mathcal{E}(?,M')|_{\mathcal{M}}$ implies that the complex $P = (\cdots \to 0 \to M_1\hat{} \to M_0\hat{} \to 0 \to \cdots)$ is quasi-isomorphic to $\mathbb{L}FM'\hat{} = \mathcal{E}(?,M')|_{\mathcal{M}}$. Therefore $\mathbb{L}FM'\hat{}$ belongs to the subcategory $\operatorname{per}\mathcal{M}$ of $\mathcal{D}\operatorname{Mod}\mathcal{M}$. Moreover, we have, for any $n \in \mathbb{Z}$ and any $M'' \in \mathcal{M}'$, the equality

$$\operatorname{Hom}_{\mathcal{D}\operatorname{Mod}\mathcal{M}}(\mathbb{L}FM'\hat{}, \mathbb{L}FM''\hat{}[n]) = \operatorname{Hom}_{\mathcal{H}^{\mathrm{b}}\operatorname{Mod}\mathcal{M}}(P, \mathcal{E}(?,M'')|_{\mathcal{M}}[n])$$

where the right–hand side vanishes for $n \neq 0, 1$. In case $n = 1$ it also vanishes, since $\operatorname{Ext}^1_{\mathcal{E}}(M', M'')$ vanishes. Now,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{H}^{\mathrm{b}}\operatorname{Mod}\mathcal{M}}(P, \mathcal{E}(?,M'')|_{\mathcal{M}}) &\simeq \operatorname{Ker}(\mathcal{E}(M_0, M'') \to \mathcal{E}(M_1, M'')) \\ &\simeq \mathcal{E}(M', M'').\end{aligned}$$

It only remains to be shown that the set $R = \{\mathbb{L}FM'\hat{},\, M' \in \mathcal{M}'\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}$. Denote by $\mathcal{R}$ the full triangulated subcategory with infinite sums of $\mathcal{D}\operatorname{Mod}\mathcal{M}$ generated by the set R. The set $\{M\hat{},\, M \in \mathcal{M}\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}$ as a triangulated category with infinite sums. Thus it is enough to show that, for any object M of $\mathcal{M}$, the complex $M\hat{}$ concentrated in degree 0 belongs to the subcategory $\mathcal{R}$. Let M be an object of $\mathcal{M}$, and let $M \overset{i}{\rightarrowtail} M'_0 \overset{p}{\twoheadrightarrow} M'_1$ be a conflation of $\mathcal{E}$ with M'_0 and M'_1 in $\mathcal{M}'$. Since $\operatorname{Ext}^1_{\mathcal{E}}(?,M)|_{\mathcal{M}}$ vanishes, we have a short exact sequence of $\mathcal{M}$-modules

$$0 \longrightarrow \mathcal{E}(?,M)|_{\mathcal{M}} \longrightarrow \mathcal{E}(?,M'_0)|_{\mathcal{M}} \longrightarrow \mathcal{E}(?,M'_1)|_{\mathcal{M}} \longrightarrow 0,$$

which yields the triangle

$$M\hat{} \longrightarrow \mathbb{L}FM_0'\hat{} \longrightarrow \mathbb{L}FM_1'\hat{} \longrightarrow \Sigma M\hat{}.$$

$\square$

As a corollary of proposition 4, up to equivalence the diagram (D) does not depend on the choice of a cluster–tilting subcategory. To be more precise: Let G be the functor which sends an object X in the category $\mathcal{H}^b(\mathcal{M}')$ to a representative of $(\mathbb{L}F)X\hat{}$ in $\mathcal{H}^b(\mathcal{M})$, and a morphism in $\mathcal{H}^b(\mathcal{M}')$ to the induced one in $\mathcal{H}^b(\mathcal{M})$.

Corollary 5. *The following diagram is commutative*

$$\begin{tikzcd}
& \mathcal{D}\,\mathrm{Mod}\ \mathcal{M}' \arrow[rr, "\mathbb{L}F"] & & \mathcal{D}\,\mathrm{Mod}\ \mathcal{M} \\
\mathcal{H}^b(\mathcal{M}') \arrow[ur, hook] \arrow[rr, "G"] \arrow[dd] & & \mathcal{H}^b(\mathcal{M}) \arrow[ur, hook] \arrow[dd] & \\
& \mathcal{H}^b(\mathcal{P}) \arrow[uu, hook] \arrow[ul, hook] \arrow[dl, hook] \arrow[rr, equal] & & \mathcal{H}^b(\mathcal{P}) \arrow[uu, hook] \arrow[ul, hook] \arrow[dl, hook] \\
\mathcal{D}^b(\mathcal{E}) \arrow[rr, equal] & & \mathcal{D}^b(\mathcal{E}) &
\end{tikzcd}$$

and the functor G is an equivalence of categories.

We denote by $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$ the full subcategory of $\mathrm{per}\,\mathcal{M}$ whose objects are the complexes with homologies in $\mathrm{mod}\,\underline{\mathcal{M}}$. The following lemma will allow us to compute the Grothendieck group of $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$ in section 2.3:

Lemma 6. *The canonical t-structure on $\mathcal{D}\,\mathrm{Mod}\ \mathcal{M}$ restricts to a t-structure on $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$, whose heart is $\mathrm{mod}\,\underline{\mathcal{M}}$.*

Proof. By [13], it is enough to show that for any object $M^\bullet$ of $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$, its truncation $\tau_{\leq 0}M^\bullet$ in $\mathcal{D}\,\mathrm{Mod}\ \mathcal{M}$ belongs to $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$. Since $M^\bullet$ is in $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$, $\tau_{\leq 0}M^\bullet$ is bounded, and is thus formed from the complexes $\mathrm{H}^i(M^\bullet)$ concentrated in one degree by taking iterated extensions. But, for any i, the $\mathcal{M}$-module $\mathrm{H}^i(M^\bullet)$ actually is an $\underline{\mathcal{M}}$-module. Therefore, by [17] (see section 1.2), it is perfect as an $\mathcal{M}$-module and it lies in $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$. □

The next lemma already appears in [21]. For the convenience of the reader, we include a proof.

Lemma 7. *The Yoneda equivalence of triangulated categories $\mathcal{H}^b(\mathcal{M}) \longrightarrow \mathrm{per}\,\mathcal{M}$ induces a triangle equivalence $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \longrightarrow \mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$.*

Proof. We first show that the cohomology groups of an $\mathcal{E}$-acyclic bounded complex M vanish on $\mathcal{P}$. Let P be a projective object in $\mathcal{E}$ and let E be a kernel in $\mathcal{E}$ of the map $M^n \longrightarrow M^{n+1}$. Since M is $\mathcal{E}$-acyclic, such an object exists, and moreover, it is an image of the map $M^{n-1} \longrightarrow M^n$. Any map from P to M^n whose composition with $M^n \to M^{n+1}$ vanishes factors through the kernel $E \rightarrowtail M^n$. Since P is projective, this factorization factors through the deflation $M^{n-1} \twoheadrightarrow E$.

$$\begin{tikzcd}
& & P \arrow[d] \arrow[drr, "0"] \arrow[dll, dashed] \arrow[ddl, dashed] & & \\
M^{n-1} \arrow[rr] \arrow[dr] & & M^n \arrow[rr] & & M^{n+1} \\
& E \arrow[ur] & & &
\end{tikzcd}$$

Therefore, we have $\mathrm{H}^n(M^{\cdot})(P) = 0$ for all projective objects P, and $\mathrm{H}^n(M^{\cdot})$ belongs to $\mathrm{mod}\,\underline{\mathcal{M}}$. Thus the Yoneda functor induces a fully faithful functor from $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ to $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$. To prove that it is dense, it is enough to prove that any object of the heart $\mathrm{mod}\,\underline{\mathcal{M}}$ of the t-structure on $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$ is in its essential image.

But this was proved in [17, section 4] (see section 1.2).

□

Proposition 8. *There is a triangle equivalence of categories*

$$\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M} \xrightarrow{\simeq} \operatorname{per}_{\underline{\mathcal{M}'}} \mathcal{M}'$$

Proof. Since the categories $\mathcal{H}^b(\mathcal{P})$ and $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ are left-right orthogonal in $\mathcal{H}^b(\mathcal{M})$, this is immediate from corollary 5 and lemma 7. □

2.3. **Grothendieck groups.** For a triangulated (resp. additive, resp. abelian) category $\mathcal{A}$, we denote by $\mathrm{K}_0^{\mathrm{tri}}(\mathcal{A})$ or simply $\mathrm{K}_0(\mathcal{A})$ (resp. $\mathrm{K}_0^{\mathrm{add}}(\mathcal{A})$, resp. $\mathrm{K}_0^{\mathrm{ab}}(\mathcal{A})$) its Grothendieck group (with respect to the mentioned structure of the category). For an object A in $\mathcal{A}$, we also denote by $[A]$ its class in the Grothendieck group of $\mathcal{A}$.

The short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \underline{\mathcal{E}} \longrightarrow 0$$

given by proposition 3 induces an exact sequence in the Grothendieck groups

$$(*) \qquad \mathrm{K}_0\left(\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})\right) \longrightarrow \mathrm{K}_0\left(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\right) \longrightarrow \mathrm{K}_0\left(\underline{\mathcal{E}}\right) \longrightarrow 0.$$

Lemma 9. *The exact sequence* $(*)$ *is isomorphic to an exact sequence*

$$(**) \qquad \mathrm{K}_0^{ab}\left(\operatorname{mod}\underline{\mathcal{M}}\right) \xrightarrow{\varphi} \mathrm{K}_0^{add}\left(\underline{\mathcal{M}}\right) \longrightarrow \mathrm{K}_0^{tri}\left(\underline{\mathcal{E}}\right) \longrightarrow 0.$$

Proof. First, note that, by [21], see also lemma 7, we have an isomorphism between the Grothendieck groups $\mathrm{K}_0\left(\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})\right)$ and $\mathrm{K}_0\left(\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\right)$. The t-structure on $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ whose heart is $\operatorname{mod}\underline{\mathcal{M}}$, see lemma 6, in turn yields an isomorphism between the Grothendieck groups $\mathrm{K}_0^{\mathrm{tri}}\left(\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\right)$ and $\mathrm{K}_0^{\mathrm{ab}}\left(\operatorname{mod}\underline{\mathcal{M}}\right)$. Next, we show that the canonical additive functor $\underline{\mathcal{M}} \xrightarrow{\alpha} \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ induces an isomorphism between the Grothendieck groups $\mathrm{K}_0^{\mathrm{add}}\left(\underline{\mathcal{M}}\right)$ and $\mathrm{K}_0^{\mathrm{tri}}\left(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\right)$. For this, let us consider the canonical additive functor $\underline{\mathcal{M}} \xrightarrow{\beta} \mathcal{H}^b(\underline{\mathcal{M}})$ and the triangle functor $\mathcal{H}^b(\mathcal{M}) \xrightarrow{\gamma} \mathcal{H}^b(\underline{\mathcal{M}})$. The following diagram describes the situation:

$$\begin{tikzcd}
\mathcal{H}^b(\underline{\mathcal{M}}) & & \mathcal{H}^b(\mathcal{M}) \arrow[ll, "\gamma"'] \arrow[d] \\
\underline{\mathcal{M}} \arrow[u, "\beta"] \arrow[r, "\alpha"] & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[ul, dashed, "\gamma"'] &
\end{tikzcd}$$

The functor γ vanishes on the full subcategory $\mathcal{H}^b(\mathcal{P})$, thus inducing a triangle functor, still denoted by γ, from $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ to $\mathcal{H}^b(\underline{\mathcal{M}})$. Furthermore, the functor β induces an isomorphism at the level of Grothendieck groups, whose inverse $\mathrm{K}_0(\beta)^{-1}$ is given by

$$\begin{array}{ccc}
\mathrm{K}_0^{\mathrm{tri}}\left(\mathcal{H}^b(\underline{\mathcal{M}})\right) & \longrightarrow & \mathrm{K}_0^{\mathrm{add}}\left(\underline{\mathcal{M}}\right) \\
{[M]} & \longmapsto & \displaystyle\sum_{i\in\mathbb{Z}}(-1)^i[M^i].
\end{array}$$

As the group $\mathrm{K}_0^{\mathrm{tri}}\left(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\right)$ is generated by objects concentrated in degree 0, it is straightforward to check that the morphisms $\mathrm{K}_0(\alpha)$ and $\mathrm{K}_0(\beta)^{-1}\,\mathrm{K}_0(\gamma)$ are inverse to each other.

□

As a consequence of the exact sequence $(**)$, we have an isomorphism between $\mathrm{K}_0^{\mathrm{tri}}\left(\underline{\mathcal{E}}\right)$ and $\mathrm{K}_0^{\mathrm{add}}\left(\underline{\mathcal{M}}\right)/\operatorname{Im}\varphi$. In order to compute $\mathrm{K}_0^{\mathrm{tri}}\left(\underline{\mathcal{E}}\right)$, the map φ has to be made explicit. We first recall some results from Iyama–Yoshino [12] which generalize results from [4]: For any indecomposable M of $\underline{\mathcal{M}}$ not in $\mathcal{P}$, there exists M^* unique up to isomorphism such that (M, M^*) is an exchange pair. This means that M and M^* are not isomorphic and that the full additive subcategory of $\mathcal{C}$ generated

by all the indecomposable objects of $\underline{\mathcal{M}}$ but those isomorphic to M, and by the indecomposable objects isomorphic to M^* is again a cluster–tilting subcategory. Moreover, $\dim \underline{\mathcal{E}}(M, \Sigma M^*) = 1$. We can thus fix two (non-split) exchange triangles

$$M^* \to B_M \to M \to \Sigma M^* \text{ and } M \to B_{M^*} \to M^* \to \Sigma M.$$

We may now state the following:

Theorem 10. *The Grothendieck group of the triangulated category $\underline{\mathcal{E}}$ is the quotient of that of the additive subcategory $\underline{\mathcal{M}}$ by all relations $[B_{M^*}] - [B_M]$:*

$$\mathrm{K}_0^{tri}\left(\underline{\mathcal{E}}\right) \simeq \mathrm{K}_0^{add}\left(\underline{\mathcal{M}}\right)/([B_{M^*}] - [B_M])_M.$$

Proof. We denote by S_M the simple $\underline{\mathcal{M}}$-module associated to the indecomposable object M. This means that $S_M(M')$ vanishes for all indecomposable objects M' in $\underline{\mathcal{M}}$ not isomorphic to M and that $S_M(M)$ is isomorphic to k. The abelian group $\mathrm{K}_0^{\mathrm{ab}}\left(\mathrm{mod}\,\underline{\mathcal{M}}\right)$ is generated by all classes $[S_M]$. In view of lemma 9, it is sufficient to prove that the image of the class $[S_M]$ under φ is $[B_{M^*}] - [B_M]$. First note that the $\mathcal{M}$-module $\mathrm{Ext}^1_{\mathcal{E}}(?, M^*)|_{\mathcal{M}}$ vanishes on the projectives ; it can thus be viewed as an $\underline{\mathcal{M}}$-module, and as such, is isomorphic to S_M. After replacing B_M and $B_{M'}$ by isomorphic objects of $\underline{\mathcal{E}}$, we can assume that the exchange triangles $M^* \to B_M \to M \to \Sigma M^*$ and $M \to B_{M^*} \to M^* \to \Sigma M$ come from conflations $M^* \rightarrowtail B_M \twoheadrightarrow M$ and $M \rightarrowtail B_{M^*} \twoheadrightarrow M^*$. The spliced complex

$$(\cdots \to 0 \to M \to B_{M^*} \to B_M \to M \to 0 \to \cdots)$$

denoted by $C^\bullet$, is then an $\mathcal{E}$-acyclic complex, and it is the image of S_M under the functor $\mathrm{mod}\,\underline{\mathcal{M}} \subset \mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M} \simeq \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$. Indeed, we have two long exact sequences induced by the conflations above:

$$0 \to \mathcal{M}(?, M) \to \mathcal{M}(?, B_{M^*}) \to \mathcal{E}(?, M^*)|_{\mathcal{M}} \to \mathrm{Ext}^1_{\mathcal{E}}(?, M)|_{\mathcal{M}} = 0 \text{ and}$$

$$0 \to \mathcal{E}(?, M^*)|_{\mathcal{M}} \to \mathcal{M}(?, B_M) \to \mathcal{M}(?, M) \to \mathrm{Ext}^1_{\mathcal{E}}(?, M^*)|_{\mathcal{M}} \to \mathrm{Ext}^1_{\mathcal{E}}(?, B_M)|_{\mathcal{M}}.$$

Since B_M belongs to $\mathcal{M}$, the functor $\mathrm{Ext}^1_{\mathcal{E}}(?, B_M)$ vanishes on $\mathcal{M}$, and the complex:

$$(C^{\hat{}}): \qquad (\cdots \to 0 \to M^{\hat{}} \to (B_{M^*})^{\hat{}} \to (B_M)^{\hat{}} \to M^{\hat{}} \to 0 \to \cdots)$$

is quasi-isomorphic to S_M.

Now, in the notations of the proof of lemma 9, $\varphi[S_M]$ is the image of the class of the $\mathcal{E}$-acyclic complex complex $C^\bullet$ under the morphism $\mathrm{K}_0(\beta)^{-1}\,\mathrm{K}_0(\gamma)$. This is $[M] - [B_M] + [B_{M^*}] - [M]$ which equals $[B_{M^*}] - [B_M]$ as claimed. $\square$

3. The generalized mutation rule

Let $\mathcal{T}$ and $\mathcal{T}'$ be two cluster–tilting subcategories of $\mathcal{C}$. Let Q and Q' be the quivers obtained from their Auslander–Reiten quivers by removing all loops and oriented 2-cycles.

Our aim, in this section, is to give a rule relating Q' to Q, and to prove that it generalizes the Fomin–Zelevinsky mutation rule.

Remark:

- Assume that $\mathcal{C}$ has cluster–tilting objects. Then it is proved in [3, Theorem I.1.6], without assuming that $\mathcal{C}$ is algebraic, that the Auslander–Reiten quivers of two cluster–tilting objects having all but one indecomposable direct summands in common (up to isomorphism) are related by the Fomin–Zelevinsky mutation rule.
- To prove that the generalized mutation rule actually generalizes the Fomin–Zelevinsky mutation rule, we use the ideas of section 7 of [9].

3.1. **The rule.** As in section 2, we fix a cluster–tilting subcategory $\mathcal{T}$ of $\mathcal{C}$, and write $\mathcal{M}$ for its preimage in $\mathcal{E}$, so that $\mathcal{T} = \underline{\mathcal{M}}$. Define Q to be the quiver obtained from the Auslander–Reiten quiver of $\underline{\mathcal{M}}$ by deleting its loops and its oriented 2-cycles. Its vertex corresponding to an indecomposable object L will also be labeled by L. We denote by a_{LN} the number of arrows from vertex L to vertex N in the quiver Q. Let $B_{\mathcal{M}}$ be the matrix whose entries are given by $b_{LN} = a_{LN} - a_{NL}$.

Let $R_{\mathcal{M}}$ be the matrix of $\langle\ ,\ \rangle_a : \mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}}) \times \mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$ in the basis given by the classes of the simple modules.

Lemma 11. *The matrices $R_{\mathcal{M}}$ and $B_{\mathcal{M}}$ are equal: $R_{\mathcal{M}} = B_{\mathcal{M}}$.*

Proof. Let L and N be two non-projective indecomposable objects in $\mathcal{M}$. Then $\dim \operatorname{Hom}(S_L, S_N) - \dim \operatorname{Hom}(S_N, S_L) = 0$ and we have:

$$\langle [S_L], [S_N] \rangle_a = \dim \operatorname{Ext}^1(S_N, S_L) - \dim \operatorname{Ext}^1(S_L, S_N) = b_{L,N}.$$

□

Let $\mathcal{T}'$ be another cluster–tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be its preimage in the Frobenius category $\mathcal{E}$. Let $(M'_i)_{i\in I}$ (resp. $(M_j)_{j\in J}$) be representatives for the isoclasses of non-projective indecomposable objects in $\mathcal{M}'$ (resp. $\mathcal{M}$). The equivalence of categories $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M} \xrightarrow{\sim} \operatorname{per}_{\underline{\mathcal{M}'}} \mathcal{M}'$ of proposition 8 induces an isomorphism between the Grothendieck groups $\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}})$ and $\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}'})$ whose matrix, in the bases given by the classes of the simple modules, is denoted by S. The equivalence of categories $\mathcal{D}\operatorname{Mod}\mathcal{M} \xrightarrow{\sim} \mathcal{D}\operatorname{Mod}\mathcal{M}'$ restricts to the identity on $\mathcal{H}^b(\mathcal{P})$, so that it induces an equivalence $\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P} \xrightarrow{\sim} \operatorname{per}\mathcal{M}'/\operatorname{per}\mathcal{P}$. Let T be the matrix of the induced isomorphism from $\mathrm{K}_0(\operatorname{proj}\mathcal{M})/\mathrm{K}_0(\operatorname{proj}\mathcal{P})$ to $\mathrm{K}_0(\operatorname{proj}\mathcal{M}')/\mathrm{K}_0(\operatorname{proj}\mathcal{P})$, in the bases given by the classes $[\mathcal{M}(?, M_j)]$, $j \in J$, and $[\mathcal{M}'(?, M'_i)]$, $i \in I$. The matrix T is much easier to compute than the matrix S. Its entries t_{ij} are given by the approximation triangles of Keller and Reiten in the following way: For all j, there exists a triangle of the form

$$\Sigma^{-1} M_j \longrightarrow \bigoplus_i \beta_{ij} M'_i \longrightarrow \bigoplus_i \alpha_{ij} M'_i \longrightarrow M_j.$$

Then, we have:

Theorem 12. a) (Generalized mutation rule) *The following equalities hold:*

$$t_{ij} = \alpha_{ij} - \beta_{ij}$$

and

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^{\mathrm{t}}.$$

b) *The category $\mathcal{C}$ has a cluster–tilting object if and only if all its cluster–tilting subcategories have a finite number of pairwise non-isomorphic indecomposable objects.*

c) *All cluster–tilting objects of $\mathcal{C}$ have the same number of indecomposable direct summands (up to isomorphism).*

Note that point c) was shown in [11, 5.3.3(1)] (see also [3, I.1.8]) and, in a more general context, in [7]. Note also that, for the generalized mutation rule to hold, the cluster–tilting subcategories do not need to be related by a sequence of mutation.

Proof. Assertions b) and c) are consequences of the existence of an isomorphism between the Grothendieck groups $\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}})$ and $\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}'})$. Let us prove the equalities a). Recall from [19, section 3.3], that the antisymmetric bilinear form

$\langle\ ,\ \rangle_a$ on $\operatorname{mod}\underline{\mathcal{M}}$ is induced by the usual Euler form $\langle\ ,\ \rangle_E$ on $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$. The following commutative diagram

$$\begin{tikzcd}[column sep=small]
\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\times\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M} \arrow[rr, "\simeq"] \arrow[dr, "\langle\,,\,\rangle_E"'] & & \operatorname{per}_{\underline{\mathcal{M}'}}\mathcal{M}'\times\operatorname{per}_{\underline{\mathcal{M}'}}\mathcal{M}' \arrow[dl, "\langle\,,\,\rangle_E"] \\
& \mathbb{Z} &
\end{tikzcd},$$

thus induces a commutative diagram

$$\begin{tikzcd}[column sep=small]
\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}})\times\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}}) \arrow[rr, "S\times S"] \arrow[dr, "\langle\,,\,\rangle_a"'] & & \mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}'})\times\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}'}) \arrow[dl, "\langle\,,\,\rangle_a"] \\
& \mathbb{Z} &
\end{tikzcd}.$$

This proves the equality $R_{\mathcal{M}} = S^{\mathrm{t}} R_{\mathcal{M}'} S$, or, by lemma 11,

$$(1) \qquad B_{\mathcal{M}} = S^{\mathrm{t}} B_{\mathcal{M}'} S.$$

Any object of $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ becomes an object of $\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}$ through the composition $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M} \hookrightarrow \operatorname{per}\mathcal{M} \twoheadrightarrow \operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}$. Let M and N be two non-projective indecomposable objects in $\mathcal{M}$. Since S_N vanishes on $\mathcal{P}$, we have

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}}\big(\mathcal{M}(?,M),S_N\big) &= \operatorname{Hom}_{\operatorname{per}\mathcal{M}}\big(\mathcal{M}(?,M),S_N\big)\\
&= \operatorname{Hom}_{\operatorname{Mod}\mathcal{M}}\big(\mathcal{M}(?,M),S_N\big)\\
&= S_N(M).
\end{aligned}$$

Thus $\dim\operatorname{Hom}_{\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}}\big(\mathcal{M}(?,M),S_N\big) = \delta_{MN}$, and the commutative diagram

$$\begin{tikzcd}[column sep=small]
\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}\times\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P} \arrow[rr, "\simeq"] \arrow[dr, "R\,\mathcal{H}\mathrm{om}"'] & & \operatorname{per}\mathcal{M}'/\operatorname{per}\mathcal{P}\times\operatorname{per}\mathcal{M}'/\operatorname{per}\mathcal{P} \arrow[dl, "R\,\mathcal{H}\mathrm{om}"] \\
& \operatorname{per}k &
\end{tikzcd},$$

induces a commutative diagram

$$\begin{tikzcd}[column sep=small]
\mathrm{K}_0(\operatorname{proj}\mathcal{M})/\mathrm{K}_0(\operatorname{proj}\mathcal{P})\times\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}}) \arrow[rr, "T\times S"] \arrow[dr, "\mathrm{Id}"'] & & \mathrm{K}_0(\operatorname{proj}\mathcal{M}')/\mathrm{K}_0(\operatorname{proj}\mathcal{P})\times\mathrm{K}_0(\operatorname{mod}\underline{\mathcal{M}'}) \arrow[dl, "\mathrm{Id}"] \\
& \mathbb{Z} &
\end{tikzcd}.$$

In other words, the matrix S is the inverse of the transpose of T:

$$(2) \qquad S = T^{\text{-t}}$$

Equalities (1) and (2) imply what was claimed, that is

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^t.$$

Let us compute the matrix T: Let M be indecomposable non-projective in $\mathcal{M}$, and let

$$\Sigma^{-1}M \longrightarrow M_1' \longrightarrow M_0' \longrightarrow M$$

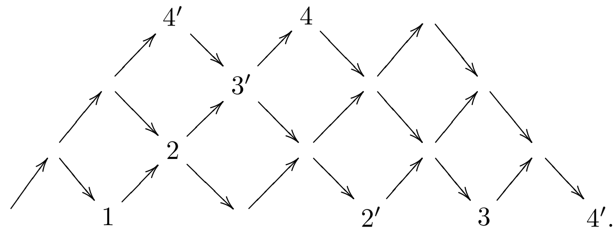
be a Keller–Reiten approximation triangle of M with respect to $\mathcal{M}'$, which we may assume to come from a conflation in $\mathcal{E}$. This conflation yields a projective resolution

$$0 \longrightarrow (M_1')\hat{} \longrightarrow (M_0')\hat{} \longrightarrow \mathcal{E}(?,M)|_{\mathcal{M}'} \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(?,M_1')|_{\mathcal{M}'} = 0.$$

so that T sends the class of $M\hat{}$ to $[(M_0')\hat{}]-[(M_1')\hat{}]$. Therefore, t_{ij} equals $\alpha_{ij}-\beta_{ij}$. $\square$

3.2. **Examples.**

3.2.1. As a first example, let $\mathcal{C}$ be the cluster category associated with the quiver of type A_4: $1 \to 2 \to 3 \to 4$. Its Auslander–Reiten quiver is the Moebius strip:



Let $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where the indecomposable M_i corresponds to the vertex labelled by i in the picture. Let also $M' = M'_1 \oplus M'_2 \oplus M'_3 \oplus M'_4$, where $M'_1 = M_1$, and where the indecomposable M'_i corresponds to the vertex labelled by i' if $i \neq 1$. One easily computes the following Keller–Reiten approximation triangles:
$\Sigma^{-1}M_1 \longrightarrow 0 \longrightarrow M'_1 \longrightarrow M_1$,
$\Sigma^{-1}M_2 \longrightarrow M'_2 \longrightarrow M'_1 \longrightarrow M_2$,
$\Sigma^{-1}M_3 \longrightarrow M'_4 \longrightarrow 0 \longrightarrow M_4$ and
$\Sigma^{-1}M_4 \longrightarrow M'_4 \longrightarrow M'_3 \longrightarrow M_4$;
so that the matrix T is given by:

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

We also have

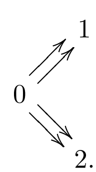
$$B_{M'} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let maple compute

$$T^{-1} B_{M'} T^{-t} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

which is B_M.

3.2.2. Let us look at a more interesting example, where one cannot easily read the quiver of M' from the Auslander–Reiten quiver of $\mathcal{C}$. Let $\mathcal{C}$ be the cluster category associated with the quiver Q:



For $i = 0, 1, 2$, let M_i be (the image in $\mathcal{C}$ of) the projective indecomposable (right) kQ-module associated with vertex i. Their dimension vectors are respectively $[1, 0, 0]$, $[2, 1, 0]$ and $[2, 0, 1]$. Let M be the direct sum $M_0 \oplus M_1 \oplus M_2$. Let M' be the direct sum $M'_0 \oplus M'_1 \oplus M'_2$, where M'_0, M'_1 and M'_2 are (the images in $\mathcal{C}$ of) the indecomposable regular kQ-modules with dimension vectors $[1, 2, 0]$, $[0, 1, 0]$

and $[2,4,1]$ respectively. As one can check, using [14], M and M' are two cluster–tilting objects of $\mathcal{C}$. To compute Keller–Reiten's approximation triangles, amounts to computing projective resolutions in $\operatorname{mod} kQ$, viewed as $\operatorname{mod}\operatorname{End}_{\mathcal{C}}(M)$. One easily computes these projective resolutions, by considering dimension vectors:
$0 \longrightarrow 8M_0 \longrightarrow M_2 \oplus 4M_1 \longrightarrow M_2' \longrightarrow 0$,
$0 \longrightarrow 2M_0 \longrightarrow M_1 \longrightarrow M_1' \longrightarrow 0$ and
$0 \longrightarrow 3M_0 \longrightarrow 2M_1 \longrightarrow M_0' \longrightarrow 0$.
By applying the generalized mutation rule, one gets the following quiver

$$\begin{tikzcd}
 & 1 \arrow[dl, "(6)"'] \\
0 \arrow[dr, "(2)"'] & \\
 & 2, \arrow[uu, "(4)"']
\end{tikzcd}$$

which is therefore the quiver of $\operatorname{End}_{\mathcal{C}}(M')$ since by [5], there are no loops or 2-cycles in the quiver of the endomorphism algebra of a cluster–tilting object in a cluster category.

3.3. **Back to the mutation rule.** We assume in this section that the Auslander–Reiten quiver of $\mathcal{T}$ has no loops nor 2-cycles. Under the notations of section 3.1, let k be in I and let (M_k, M_k') be an exchange pair (see section 2.3). We choose $\underline{\mathcal{M}}'$ to be the cluster-tilting subcategory of $\mathcal{C}$ obtained from $\underline{\mathcal{M}}$ by replacing M_k by M_k', so that $M_i' = M_i$ for all $i \neq k$. Recall that T is the matrix of the isomorphism $\mathrm{K}_0(\operatorname{proj}\mathcal{M})/\,\mathrm{K}_0(\operatorname{proj}\mathcal{P}) \longrightarrow \mathrm{K}_0(\operatorname{proj}\mathcal{M}')/\,\mathrm{K}_0(\operatorname{proj}\mathcal{P})$.

Lemma 13. *Then, the (i,j)-entry of the matrix T is given by*

$$t_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}|+b_{ij}}{2} & \textit{if } j = k \\ \delta_{ij} & \textit{else.} \end{cases}$$

Proof. Let us apply theorem 12 to compute the matrix T. For all $j \neq k$, the triangle $\Sigma^{-1}M_j \to 0 \to M_j' = M_j$ is a Keller–Reiten approximation triangle of M_j with respect to $\mathcal{M}'$. We thus have $t_{ij} = \delta_{ij}$ for all $j \neq k$. There is a triangle unique up to isomorphism

$$M_k' \longrightarrow B_{M_k} \longrightarrow M_k \longrightarrow \Sigma M_k'$$

where $B_{M_k} \longrightarrow M_k$ is a right $\mathcal{T} \cap \mathcal{T}'$-approximation. Since the Auslander–Reiten quiver of $\mathcal{T}$ has no loops and no 2-cycles, B_{M_k} is isomorphic to the direct sum: $\bigoplus_{i \in I} (M_j')^{a_{ik}}$. We thus have $t_{ik} = -\delta_{ik} + a_{ik}$, which equals $\frac{|b_{ik}|+b_{ik}}{2}$. Remark that, by lemma 7.1 of [9], as stated in section 1.1, we have $T^2 = Id$, so that $S = T^{\mathrm{t}}$ and

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}|-b_{ij}}{2} & \text{if } i = k \\ \delta_{ij} & \text{else.} \end{cases}$$

□

Theorem 14. *The matrix $B_{\mathcal{M}'}$ is obtained from the matrix $B_{\mathcal{M}}$ by the Fomin–Zelevinski mutation rule in the direction M.*

Proof. By [2] (see section 1.1), and by lemma 13, we know that the mutation of the matrix $B_{\mathcal{M}}$ in direction M is given by $TB_{\mathcal{M}'}T^{\mathrm{t}}$, which is $B_{\mathcal{M}}$, by the generalized mutation rule (theorem 12). □

3.4. **Cluster categories.** In [1], the authors study the Grothendieck group of the cluster category $\mathcal{C}_A$ associated to an algebra A which is either hereditary or canonical, endowed with any admissible triangulated structure. A triangulated structure on the category $\mathcal{C}_A$ is called admissible in [1] if the projection functor from the bounded derived category $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ to $\mathcal{C}_A$ is exact (triangulated). They define a Grothendieck group $\overline{\mathrm{K}}_0(\mathcal{C}_A)$ with respect to the triangles induced by those of $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$, and show that it coincides with the usual Grothendieck group of the cluster category in many cases:

Theorem 15. [Barot–Kussin–Lenzing] *We have* $\mathrm{K}_0(\mathcal{C}_A) = \overline{\mathrm{K}}_0(\mathcal{C}_A)$ *in each of the following three cases:*

(i) *A is canonical with weight sequence $(p_1, \ldots, p_t)$ having at least one even weight.*
(ii) *A is tubular,*
(iii) *A is hereditary of finite representation type.*

Under some restriction on the triangulated structure of $\mathcal{C}_A$, we have the following generalization of case (iii) of theorem 15:

Theorem 16. *Let A be a finite-dimensional hereditary algebra, and let $\mathcal{C}_A$ be the associated cluster category with its triangulated structure defined in* [16]*. Then we have* $\mathrm{K}_0(\mathcal{C}_A) = \overline{\mathrm{K}}_0(\mathcal{C}_A)$.

Proof. By lemma 3.2 in [1], this theorem is a corollary of the following lemma. □

Lemma 17. *Under the assumptions of section 3.1, and if moreover $\underline{\mathcal{M}}$ has a finite number n of non-isomorphic indecomposable objects, then we have an isomorphism* $\mathrm{K}_0(\mathcal{C}) \simeq \mathbb{Z}^n / \operatorname{Im} B_{\mathcal{M}}$.

Proof. This is a restatement of theorem 10. □

References

[1] M. Barot, D. Kussin, and H. Lenzing. The Grothendieck group of a cluster category. *J. Pure Appl. Algebra*, 212(1):33–46, 2008.

[2] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[3] Aslak Bakke Buan, Osamu Iyama, Idun Reiten, and Jeanne Scott. Cluster structures for 2-Calabi–Yau categories and unipotent groups. *preprint arXiv: math/0701557[math.RT].*

[4] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov. Tilting theory and cluster combinatorics. *Adv. Math.*, 204(2):572–618, 2006.

[5] Aslak Bakke Buan, Robert J. Marsh, and Idun Reiten. Cluster mutation via quiver representations. *Comment. Math. Helv.*, 83(1):143–177, 2008.

[6] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras. *to appear in Invent.Math. arXiv: math/0506018[math.RT].*

[7] Raika Dehy and Bernhard Keller. On the combinatorics of rigid objects in 2-Calabi–Yau categories. *to appear in Int. Math. Res. Not. preprint arXiv: math/0709.0882[math.RT].*

[8] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[9] Christof Geiß, Bernard Leclerc, and Jan Schröer. Rigid modules over preprojective algebras. *Invent. Math.*, 165(3):589–632, 2006.

[10] Dieter Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[11] Osamu Iyama. Auslander correspondence. *Adv. Math.*, 210(1):51–82, 2007.

[12] Osamu Iyama and Yuji Yoshino. Mutation in triangulated categories and rigid Cohen–Macaulay modules. *to appear in Invent. Math. preprint arXiv: math/0607736[math.RT].*

[13] B. Keller and D. Vossieck. Aisles in derived categories. *Bull. Soc. Math. Belg. Sér. A*, 40(2):239–253, 1988. Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987).

[14] Bernhard Keller. Quiver mutation in Java. *Java applet available at B. Keller's home page.*

[15] Bernhard Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.

[16] Bernhard Keller. On triangulated orbit categories. *Doc. Math.*, 10:551–581 (electronic), 2005.

[17] Bernhard Keller and Idun Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi–Yau. *Adv. Math.*, 211(1):123–151, 2007.

[18] Bernhard Keller and Dieter Vossieck. Sous les catégories dérivées. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(6):225–228, 1987.

[19] Yann Palu. Cluster characters for triangulated 2-Calabi–Yau categories. *To appear in Ann. Inst. Fourier, preprint arXiv: math/0703540v2 [math.RT].*

[20] Jeremy Rickard. Derived equivalences as derived functors. *J. London Math. Soc. (2)*, 43(1):37–48, 1991.

[21] Gonçalo Tabuada. On the structure of Calabi-Yau categories with a cluster tilting subcategory. *Doc. Math.*, 12:193–213 (electronic), 2007.

[22] Andrei Zelevinsky. What is ... a cluster algebra? *Notices Amer. Math. Soc.*, 54(11):1494–1495, 2007.

Université Paris 7 - Denis Diderot, UMR 7586 du CNRS, case 7012, 2 place Jussieu, 75251 Paris Cedex 05, France.

E-mail address: `palu@math.jussieu.fr`